to angular acceleration by using the configuration of the accelerometer pair.

The results of the simulations are shown in Fig. 3. It shows the true θ and the estimate $\hat{\theta}$ of the head orientation, together with the sampled signal θ_T. It can be seen that the error between the true head orientation θ and the estimated one $\hat{\theta}$ diminishes to about 10% of θ. This level of the residual error is comparable to the measurement noise of the HMD measurement system and is adequate for the required image stabilization on the HMD. It also demonstrates that the phase shift between the true θ and the estimated head motion $\hat{\theta}$ is almost eliminated. This is in contrast to the large phase shift between the sampled signal θ_T of the head measurement system and the true head motion θ.

The effect of the finite resolution of the head position and orientation measurement device on the performance of the complementary filter was examined, and it is displayed in Fig. 4. The true head motion was simulated as a low frequency of 0.1 Hz and a small amplitude of the 0.5-deg signal. As a result of the finite resolution of the measurement system, its output is a staircase signal Θ_T, and the error between the two signals is a jagged signal $\tilde{\theta}_T$. The output of the complementary filter, however, is a smooth signal resembling the true head motion although with a slightly larger amplitude. For this case, it should be pointed out that the input of the accelerometer to the filter is zero since the head acceleration is below the threshold of the accelerometer, as mentioned earlier. However, since in this low-frequency range the phase shift due to the sampling process is very low (less than 2 deg), the acceleration input is not essential.

Conclusions

The method based on complementary filtering has been shown to be effective in compensating for the image stabilization error due to sampling delays of helmet-mounted display position and orientation measurements. These delays would have otherwise prevented the stabilization of the image in helmet-mounted displays. In addition, the method has been shown to improve the resolution of the head orientation measurement, especially at low frequencies, hence providing smoother head control commands, which are essential for precise head pointing and teleoperation.

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True Anomaly Approximation for Elliptical Orbits

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Introduction

THIS Note describes a direct technique for determining true anomaly f on an elliptic orbit at a specified time, without the intermediate step of solving Kepler's equation. The accuracy of the technique is best for small eccentricities e. At $e = 0.7$, the maximum error in f is 0.6 degree of arc; for e less than or equal to 0.27, the errors are less than 1 min of arc; and for e less than 0.06, they are smaller than 1 s of arc.

A direct computation of f from mean anomaly M is desirable because fewer computations are necessary, as compared to the usual procedure involving eccentric anomaly E as an intermediate variable. No satisfactory methods to accomplish this are known for large values of e; however, the Fourier-Bessel series[1] is useful for values of e less than about 0.5

$$\begin{aligned} f = M &+ \left[2e - (1/4)e^3 + (5/96)e^5\right] \sin M \\ &+ \left[(5/4)e^2 - (11/24)e^4 + (17/192)e^6\right] \sin 2M \\ &+ \left[(13/12)e^3 - (43/64)e^5\right] \sin 3M \\ &+ \left[(103/96)e^4 - (451/480)e^6\right] \sin 4M \\ &+ (1097/960)e^5 \sin 5M + (1223/960)e^6 \sin 6M + \mathcal{O}(e^7) \end{aligned} \tag{1}$$

It was found that the accuracy of the present algorithm is comparable to that of Eq. (1) to $\mathcal{O}(e^4)$ for small e and is much better than even $\mathcal{O}(e^6)$ for large e.

The present algorithm is important for another reason. It is derived from considerations of elliptic orbit motion as described by the methods of velocity space.[2] Normally, time is not brought naturally into the discussions of these techniques. Usually, the concern is with orbit parameters (such as velocity) at different spatial (rather than time) positions of the orbit. If the time could be brought into the velocity space formalism with elegance that compares to that of the spatial description, than a truly powerful mathematical method would result. The present Note is a first step toward that end.

Velocity Space Foundation

As described by Abelson et al.[2] orbits that are elliptical in terms of displacement vectors become circular in terms of velocity vectors. The center of the velocity circle of invariant radius u is offset by an invariant vector $\boldsymbol{z}$ with respect to the velocity space origin. If one imagines planetary motion about the center of the sun, the vector $\boldsymbol{u}$ is at each moment perpendicular to the planet's position vector $\boldsymbol{r}$. The magnitude of $\boldsymbol{u}$ is inversely proportional to the planet's angular momentum through the force constant. Additionally, the magnitude of vector z is such that $z = eu$. For the present work, normalization is employed ($u = 1$). There are three points of outstanding significance in the normalized plot. At $f = 0$, the velocity mag-

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nitude is $1+e$. At $f=\pi$, it is $1-e$. At f such that $\cos f=-e$, the velocity is $(1-e^2)^{1/2}$, which corresponds to eccentric anomaly, $E=\pi/2$, and mean anomaly, $M=\pi/2-e$. For this special case, the anomalies f and E are defined with respect to the center and origin 0, respectively, of the normalized velocity circle.

Algorithm Considerations

Based on considerations of velocity space graphics, a vector of constant direction and variable magnitude L was chosen such that

$$L = \cos M - \sin M/\tan f \tag{2}$$

At the three particular values of $M=0$, $\pi/2-e$, and π; L and its derivative with respect to M are determined by e as follows:

$$L_0 = 1-(1-e)\left[(1-e)/(1+e)\right]^{1/2} \tag{3}$$

$$L_{(\pi/2-e)} = \sin e + e\cos e/(1-e^2)^{1/2} \tag{4}$$

$$L_\pi = -1+(1+e)\left[(1+e)/(1-e)\right]^{1/2} \tag{5}$$

$$\left(\frac{dL}{dM}\right)_{(\pi/2-e)} = -\cos e + (e\sin e + \cos e)/(1-e^2)^{1/2} \tag{6}$$

$$\left(\frac{dL}{dM}\right)_0 = \left(\frac{dL}{dM}\right)_\pi = 0 \tag{7}$$

A power series approximation for L in terms of M was used,

$$L = L_0 + a_1M + a_2M^2 + a_3M^3 + a_4M^4 + a_5M^5 \tag{8}$$

where the constraints implicit in Eqs. (3-7) were invoked. The zero slope at $M=0$ requires that a_1 vanish. The other constraints uniquely determine the a_2–a_5 coefficients as a function of e alone. The details are given in the Appendix. Operationally, the coefficients need to be evaluated only once for an orbit of assumed constant eccentricity. Thereafter, the determination of true anomaly is simple and speedy through the relationship:

$$f = \tan^{-1}\left[\sin M/(\cos M - L)\right]$$

with

$$L = L_0 + M^2\left\{a_2 + M\left[a_3 + M(a_4 + Ma_5)\right]\right\} \tag{9}$$

For example, consider the orbit of Mercury, $e=0.205627$. The coefficients are $L_0=0.355192$, $a_2=3.78717\times10^{-2}$, $a_3=-3.23410\times10^{-3}$, $a_4=-2.77554\times10^{-3}$, and $a_5=4.14826\times10^{-4}$. Using these values in Eq. 9, the true anomaly may be easily estimated to a radian accuracy of four significant figures, assuming no uncertainty in e.

Accuracy of Results

The power series estimate for L [Eq. (8)] works well because the shape of the L vs M curve is not highly sensitive to e. The range of variation in L gets quite large as e approaches 1; however, the character of the curve is generally retained over a considerable range of e.

Table 1 provides maximum errors in the estimated f for different values of e. For $e>0.7$ the errors build rapidly, and the algorithm becomes unsatisfactory. For comparison, the accuracy of the Fourier-Bessel series is also given for cases in which the series is terminated at e^4 and e^6, respectively. It can be seen that the present algorithm is superior for all cases of $e>0.3$ and is increasingly so at large e. It is of the same accuracy as Fourier-Bessel $\mathcal{O}(e^4)$ for $e<0.1$, and it requires a comparable number of computations (once the coefficients have been determined).

Table 1 Errors of present algorithm vs eccentricity compared to Fourier-Bessel series[a]

		Fourier-Bessel series	
e	Present algorithm	$[\mathcal{O}(e^4)]$	$[\mathcal{O}(e^6)]$
0.10	0.050	0.066	0.0012
0.20	0.40	2.0	0.11
0.30	1.4	14.0	2.0
0.40	2.8	61.0	14.0
0.50	3.6	170.0	64.0
0.60	5.3	410.0	230.0
0.70	36.0	880.0	600.0
0.80	140.0	1700.0	1600.0

[a]Errors are absolute value maxima in minutes of arc.

Appendix

To evaluate the coefficients a_2–a_5, the constraints yield

$$L_\pi - L_0 = \Delta L_2 = a_2\pi^2 + a_3\pi^3 + a_4\pi^4 + a_5\pi^5 \tag{A1}$$

$$L_{(\pi/2-e)} - L_0 = \Delta L_1 = a_2M_c^2 + a_3M_c^3 + a_4M_c^4 + a_5M_c^5 \tag{A2}$$

where $M_c=\pi/2-e$

$$\left(\frac{dL}{dM}\right)_{(\pi/2-e)} = S = a_2 2M_c + a_3 3M_c^2 + a_4 4M_c^3 + a_5 5M_c^4 \tag{A3}$$

$$\left(\frac{dL}{dM}\right)_\pi = 0 = a_2 2\pi + a_3 3\pi^2 + a_4 4\pi^3 + a_5 5\pi^4 \tag{A4}$$

In terms of the known parameters ΔL_1, ΔL_2, and S, Eqs. (A1-A4) were solved for the four unknown coefficients a_2–a_5 using conventional methods of linear algebra.

Using $D=[\pi M_c(\pi-M_c)]^4$

$$\begin{aligned} Da_2 = {} & \pi^2M_c^6\,\Delta L_2(5\pi^2 - 8\pi M_c + 3M_c^2) \\ & + \pi^6M_c^2\,\Delta L_1(5M_c^2 + 3\pi^2 - 8\pi M_c) \\ & - \pi^6M_c^3S(\pi - M_c)^2 \end{aligned} \tag{A5}$$

$$\begin{aligned} Da_3 = {} & \pi M_c^5\,\Delta L_2(12\pi^2M_c - 10\pi^3 - 2M_c^3) \\ & + \pi^5M_c\Delta L_1(12\pi M_c^2 - 10M_c^3\pi^3 - 2\pi^3) \\ & + \pi^5M_c^2S(2M_c^3 + \pi^3 - 3\pi M_c^2) \end{aligned} \tag{A6}$$

$$\begin{aligned} Da_4 = {} & \pi M_c^4\,\Delta L_2(5\pi^3 - 9\pi M_c^2 + 4M_c^3) \\ & + \pi^4M_c\Delta L_1(5M_c^3 + 4\pi^3 - 9\pi^2M_c) \\ & + \pi^4M_c^2S(3\pi^2M_c - M_c^3 - 2\pi^3) \end{aligned} \tag{A7}$$

$$\begin{aligned} Da_5 = {} & \pi M_c^4\,\Delta L_2(6\pi M_c - 4\pi^2 - 2M_c^2) \\ & + \pi^4M_c\Delta L_1(6\pi M_c - 4M_c^2 - 2\pi^2) \\ & + \pi^4M_c^2S(\pi - M_c)^2 \end{aligned} \tag{A8}$$

Although Eqs. (A5-A8) correspond to a considerable amount of computation, it should be noted that the coefficients are a function only of e. In other words, for an object possessing constant eccentricity, they must be evaluated only once. For application to osculating element algorithms, they are probably amenable to straightforward table lookup and interpolation.

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